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**Analytical derivation and numerical validation of the functionally graded arches under pressure and thermal loadings**

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**Abstract**

The functionally graded materials (FGM) were developed originally to resist the high temperature of aircraft in the 1980s. With the development and improvement of the manufacturing technology, more and more FGM types have been used and studied in practical engineering applications. The present work focuses on the stability performance of the heated FGM arch subjected to the pressure field. The material properties distribute non-uniformly in the thickness direction of the FGM arch, which may show different buckling mechanisms from the conventional uniform materials. It is found the arch expands when the thermal field is introduced. The uniform pressure is applied to the arch radially inward, resulting in asymmetric or symmetric deformations of the arch. The pressure capacity is evaluated theoretically and expressed explicitly based on the minimum potential energy principle. To verify the above analytical solution, a simulated model is developed numerically. The numerical pressure capacity is compared successfully with the analytical one. In addition, the present theoretical and simulated results are further validated by other closed-form expressions. Finally, the effect of geometric and material parameters on the stability behavior of the FGM arch are investigated and discussed.

**Introduction**

Arch-shaped structures are characterized by a high slenderness ratio, which may result in instability issues. The buckling behavior of the traditional arch with uniform materials was sufficiently discussed by Timoshenko and Gere [1], Simitses and Hodges [2], and Karnovsky [3]. In the present investigation, the arch with functionally graded materials (FGM) will be discussed for its collapse performance. It is found the non-uniform distributions of the material properties of the FGM arch results in different buckling mechanisms from the arch with homogeneous materials.

**Literature Review**

The FGM was developed originally to help the aircraft to resist the heat [4]. Except for the high thermal performance, the mechanical behavior of the FGM is also excellent to be used in the nuclear power plants, and the deep-sea apparatus [5]. Asgari et al. [6] studied the instability of the FGM arches subjected to a pure thermal field. It was found the material may fail before the occurrence of the thermal buckling. Bateni and Eslami [7] highlighted the buckling of the FGM arches under a point load. The buckling load may be obtained by solving a set of governing equations. Kiss [8] evaluated the effect of loading imperfection on the FGM arches. The results indicated that the buckling behavior was sensitive to the load position. Lim et al. [9] reported the in-plane vibration of FGM arches in a thermal field. It was concluded that the temperature variation may result in a significant effect on the vibration behavior of the FGM arch. George and Charalampakis [10] optimized the natural frequencies of the FGM arches by presenting a new methodology. Pi and Bradford [11] studied the buckling of a uniform arch subjected to pressure and thermal loadings. It was found the buckling pressure was sensitive to the variational thermal field.

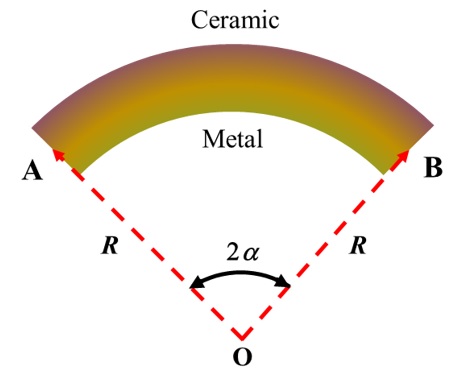


Figure 1: Material distributions of an FGM arch

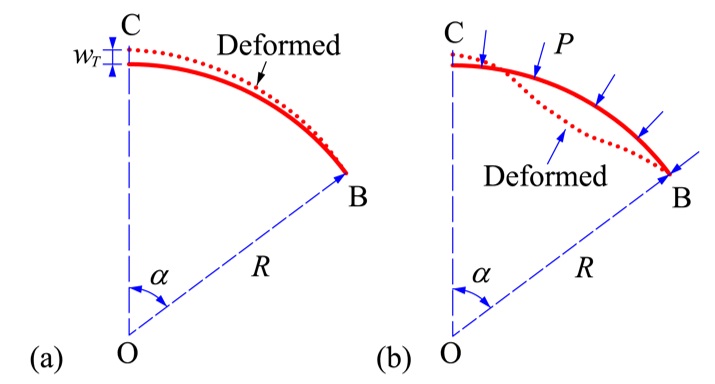


Figure 2: Deformed shapes of an FGM arch under (a) a temperature rise field, and (b) combined temperature rise field and pressure field

An FGM arch is depicted in Figure 1. The ceramic material is rich in the vicinity of the outer surface, and the metal material is rich in the vicinity of the inner surface, respectively. An increase in the temperature may result in a displacement () as shown in Figure 2(a). The pressure field is introduced after the thermal field. Based on the deformation in Figure 2(b), the present paper is concerned with the stability of the FGM arches under pressure and thermal fields. The analytical buckling pressure is derived and verified by numerical results. Finally, the effects of geometric and material parameters are discussed.

**Assumptions**

The displacement components are shown in Figure 3. The central angle is and the half-length of the arch is , where . The width is , the thickness is , and the radius is , respectively. There are two hypotheses assumed. The first one is only the in-plane buckling is considered and the second one is the materials are elastic and temperature-dependent.

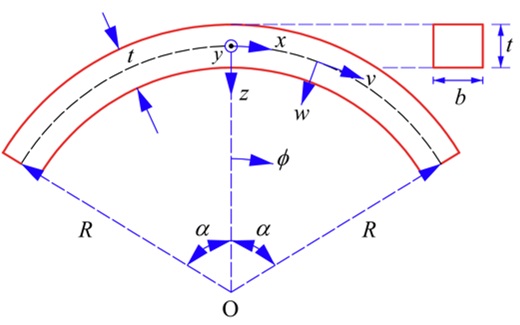


Figure 3: Displacement components of an FGM arch

**Derivation of Buckling Pressure**

As shown in Figure 3, displacement is defined as and represent the radial and the tangential displacements, respectively. Following the thin-walled shell theory [12], the displacement components are expressed into

(1)

(2)

where is the tangential displacement at the mid-axis and is the radial displacement at the mid-axis. As shown in Figure 1, the volume ratios of the ceramic and metal are expressed quantitatively as [13]

and (3)

where () is the volume ratio exponent. Following the rule of mixture, Young’s modulus and coefficient of thermal expansion are expressed as

(4a)

(4b)

where () is Young’s modulus of the ceramic (metal), and () is the coefficient of thermal expansion of the ceramic (metal), respectively. The total strain are expressed by

(5a)

(5b)

where is the circumferential strain at the mid-axis and is the strain from the bending curvature, respectively. The circumferential strain induced by the temperature variation () is

(6)

Therefore, the circumferential stress takes the form

(7)

Similarly, the bending curvature yields

(8)

Now the total potential energy function is integrated by

(9)

where is the volume of the half arch, and

(10)

where is the work generated by the pressure, and is the pressure, respectively.

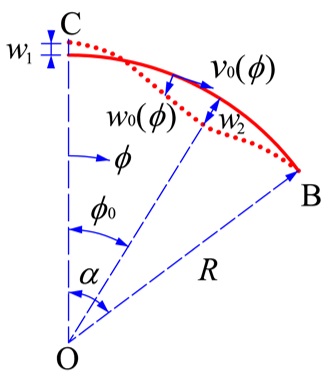


Figure 4: Displacement parameters of the deformed FGM arch

The deformed shape of the FGM arch is shown in Figure 4. Half arch is studied due to symmetry. The radial displacement may be described by [14, 15]

(11a)

The rotation and bending curvature are calculated by

(11b)

(11c)

The curvature is continuous at , yielding

(12)

where . Here, the circumferential strain is averaged to obtain the equilibrium path, yielding

(13)

where is the averaged circumferential strain. The following integration vanishes

(14)

since at the crown and at the right ends. Introducing Eqs. (11), (12) and (14) to Eq. (13) yields

(15a)

where

(15b)

(15c)

After replacing by in Eq. (9), the energy function is simplified into

(16)

where is the averaged circumferential force, and written by

(17a)

(17b)

Here, , , are the stretching, stretching-bending, and bending stiffness respectively, and , are the thermal circumferential force and bending moment, yielding

(18a)

(18b)

(18c)

(18d)

(18e)

Taking the first derivative of the total potential energy function to zero, yielding

(19)

Simultaneously, the equilibrium path satisfies

(20)

where is the second derivative of the energy function. Two governing equations are obtained from Eq. (19), yielding

(21a)

(21b)

Substituting Eqs. (21a) into (21b), the averaged circumferential force yields

(22a)

where

(22b)

Introducing Eq. (17a) to Eq. (22a), a quadratic formula is obtained for , yielding

(23)

Eq. (23) yields two roots

(24)

The smaller positive root satisfies Eq. (20). By substituting Eq. (24) into Eq. (12), the crown displacement is expressed explicitly as

(25)

Introducing Eq. (22a) and Eq. (24) into Eq. (21a), the pressure is expressed as

(26)

Newton-Raphson iteration will be used to obtain and the critical buckling pressure . The specific iteration is as follows: defining an initial , then substituting into Eq. (28), if , then define , if , then define …repeating the above process until to find and its corresponding . Substituting into Eq. (26), the critical buckling pressure for the arches is

(27)

where , , and correspond to .

**Numerical Model**

ABAQUS software [16] is used for the simulation. The material properties can be expressed quantitatively into

(28)

where , , , and are constant as shown in Table 1 [17]. The radius of the arch and the thickness , respectively. There are twenty layers in the thickness direction, indicating the thickness of each layer is 0.00025 m. In the tangential direction, the mesh size is 0.0005 mm. The arch is discrete with 9600 elements for the case as shown in Figure 5(a), where  represents eight-node reduced-integration plane stress elements. The temperature rise () is applied before the pressure loading as shown in Figure 8(b).

Table 1: Material properties of SUS304 and Si3N4

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Material | Properties |  |  |  |  |  |
| SUS304 |  | 0 |  |  | 0 | 0 |
|  | 0 |  |  |  | 0 |
| Si3N4 |  | 0 |  |  | 0 | 0 |
|  | 0 |  |  |  |  |

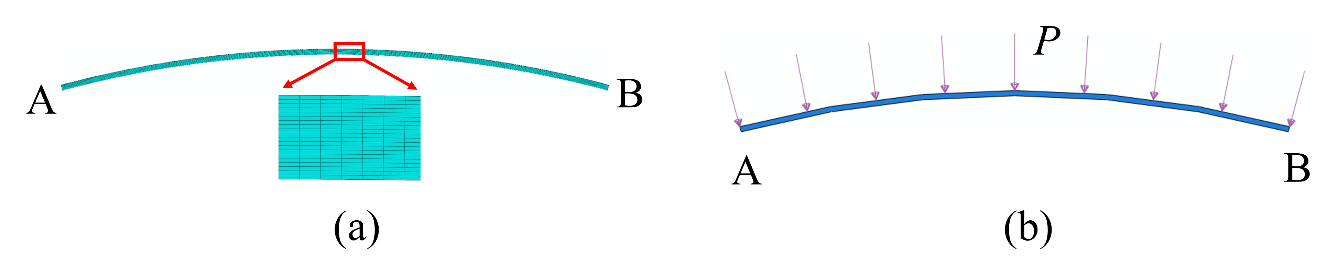


Figure 5: (a) Mesh, and (b) boundaries of an FGM arch

**Comparisons**

The numerical results are compared with the analytical solutions of Eq. (27). One may observe the numerical buckling pressure are in good agreement with the analytical solutions, and the maximum difference is no more than 5% for all examined cases with two temperature rises, four different central angles, and five different volume fraction exponents, respectively.

Table 2: The comparison of critical buckling pressure between present analytical and numerical results

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  | FEM ()  (MPa) | Eq. (27) ()  (MPa) |  | |
| 200 |  | 0 | 1.97 | 1.98 | 0.00 |
|  |  | 0.5 | 1.74 | 1.74 | 0.00 |
|  |  | 1 | 1.65 | 1.66 | 0.00 |
|  |  | 2 | 1.59 | 1.59 | 0.00 |
|  |  | 5 | 1.54 | 1.52 | 0.01 |
|  |  | 0 | 0.711 | 0.711 | 0.00 |
|  |  | 0.5 | 0.623 | 0.622 | 0.00 |
|  |  | 1 | 0.591 | 0.591 | 0.00 |
|  |  | 2 | 0.568 | 0.567 | 0.00 |
|  |  | 5 | 0.546 | 0.540 | 0.01 |
|  |  | 0 | 0.367 | 0.362 | 0.02 |
|  |  | 0.5 | 0.321 | 0.316 | 0.02 |
|  |  | 1 | 0.305 | 0.300 | 0.01 |
|  |  | 2 | 0.292 | 0.288 | 0.02 |
|  |  | 5 | 0.281 | 0.274 | 0.03 |
|  |  | 0 | 0.228 | 0.219 | 0.04 |
|  |  | 0.5 | 0.199 | 0.191 | 0.05 |
|  |  | 1 | 0.188 | 0.181 | 0.04 |
|  |  | 2 | 0.182 | 0.174 | 0.05 |
|  |  | 5 | 0.174 | 0.165 | 0.05 |
| 400 |  | 0 | 1.99 | 2.06 | 0.03 |
|  |  | 0.5 | 1.74 | 1.78 | 0.02 |
|  |  | 1 | 1.65 | 1.68 | 0.02 |
|  |  | 2 | 1.59 | 1.60 | 0.01 |
|  |  | 5 | 1.53 | 1.52 | 0.01 |
|  |  | 0 | 0.701 | 0.726 | 0.03 |
|  |  | 0.5 | 0.607 | 0.619 | 0.02 |
|  |  | 1 | 0.571 | 0.582 | 0.02 |
|  |  | 2 | 0.546 | 0.554 | 0.01 |
|  |  | 5 | 0.523 | 0.521 | 0.00 |
|  |  | 0 | 0.359 | 0.367 | 0.02 |
|  |  | 0.5 | 0.310 | 0.312 | 0.01 |
|  |  | 1 | 0.291 | 0.293 | 0.01 |
|  |  | 2 | 0.278 | 0.278 | 0.00 |
|  |  | 5 | 0.266 | 0.262 | 0.02 |
|  |  | 0 | 0.222 | 0.221 | 0.00 |
|  |  | 0.5 | 0.192 | 0.188 | 0.02 |
|  |  | 1 | 0.180 | 0.176 | 0.02 |
|  |  | 2 | 0.173 | 0.167 | 0.03 |
|  |  | 5 | 0.164 | 0.157 | 0.04 |

One more comparison is performed between the present analytical and numerical results with the prediction by Pi and Bradford [11] for the homogeneous arch (). The arch material is ceramic, and the material properties depend on Eq. (27). The comparison results are depicted in Figure 12. It is observed the present analytical and numerical results agree well with the prediction from Pi and Bradford [11] for the two temperature variations.

(a) (b)

Figure 6: Comparisons between the present analytical and numerical results, as well as the prediction from Pi and Bradford [11] for (a) and (b)

**Parametric Analyses**

Figure 7 describes the effects of the volume fraction exponent on the buckling pressure with . The buckling pressure decreases with the increase of . This is because Young’s modulus reduces when increases. Multiple equilibrium paths are observed as shown in the report of Pi et al. [18]. The equilibrium paths with four varied temperatures are plotted in Figure 8. It is found the displacement starts from a negative value in the horizontal axis due to the thermal effect. Furthermore, the buckling pressure varies slightly with the temperature rise because the temperature rise shows two effects on the arch: on one hand, the temperature rise increases the radially-outward displacement, which is beneficial to the buckling pressure; on the other hand, the temperature rise reduces Young’s modulus, which is unbeneficial to the buckling pressure. Therefore, the buckling pressure may be nonlinear with the temperature rise.

Figure 9 depicts the distributions of bending moment through the arch span when the critical buckling occurs. It is seen the maximum positive bending moment occurs on the crown (mid-span), while the maximum negative bending moment occurs on the position between mid-span and edge, corresponding to the position with the maximum radial displacement. Furthermore, a higher temperature rise increases the bending moment. Besides, the bending moment is nonlinear to the volume fraction exponent . This is because the thermal moment generates due to the non-uniform distribution of material properties in the cross-section.

(a)  (b) 

(c)  (d) 

Figure 7: The equilibrium paths of clamped-clamped FGM arches with different volume fraction exponents

(a)  (b) 

(c)  (d) 

Figure 8: The equilibrium paths of the FGM arches under different temperature rises

(a) (b) 

Figure 9: The bending moment of the clamped-clamped arch with different temperature rises

The distributions of the hoop force (circumferential force) are shown in Figure 10. It is found the hoop force decreases with the increase of , and a higher temperature rise results in a lower hoop force. This is because the increase of volume fraction exponent and temperature rise will reduce Young’s modulus. A few fluctuations of the hoop force are observed, and these fluctuations are neglected in Eq. (17). The hoop force is simplified as constant. Such simplification results in an insignificant difference in the buckling pressure as shown in Table 2. Figure 11 illustrates the distribution of hoop strain and stress in the thickness direction of the mid-span (). It is seen the hoop strain is distributed linearly in the thickness direction. Generally, the higher the temperature, the higher the hoop strain and stress. However, the stress is distributed nonlinearly due to the non-symmetrical dispersion of Young’s modulus in the thickness direction (). For a homogeneous arch, the stress is linear in the thickness direction.

(a) (b)

Figure 10: The hoop force of the clamped-clamped arch with different temperature rises

(a)(b) 

(c)  (d)

Figure 11: Distribution of mid-span strain and stress in the thickness

**Conclusions**

The stability mechanism of the FGM arch is studied. Based on the present analytical and numerical results, several main conclusions are drawn:

* The derived analytical buckling pressure is verified successfully by numerical results. For an arch with homogeneity, the present analytical and numerical results are verified successfully by other closed-form expressions.
* The strain and stress of the FGM arch have different distributions from the homogeneous arch due to the non-symmetrical material properties in the cross-section.
* The critical buckling pressure is not linear with the temperature rise field.

**Future work**

Based on the present research work, the following research may be developed later:

* Optimization of the material distribution in the radial and tangential directions of the FGM arch to sustain the highest buckling pressure.
* Experimental work is necessary to verify the present analytical and numerical studies.
* Examination of the responses of the FGM arch under other mechanical and/or thermal fields.

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**Biographies**

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